① We want a power series solution for

$$y'' - (x+1)y' + x^2 y = 0$$
$$y'(0) = 1, \quad y(0) = 1$$

Here $x_0 = 0$

We have

$$a_1(x) = -(x+1) = -1 - x$$
$$a_2(x) = x^2$$
$$b(x) = 0$$

these are already in power series form centered at $x_0 = 0$. They all have radius of convergence $r = \infty$

Thus, there must be a unique solution to the initial-value problem centered at $x_0 = 0$ of the form

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n$$
$$= y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots$$

So,

$$y'' = (x+1)y' - x^2 y$$
$$y(0) = 1$$
$$y'(0) = 1$$
$$y''(0) = (0+1)\underbrace{y'(0)}_{1} - 0^2\underbrace{y(0)}_{1} = 1$$

$y(0) = 1$
$y'(0) = 1$
$y''(0) = 1$

Differentiate $y'' = (x+1)y' - x^2 y$ to get the next step.

$$y''' = y' + (x+1)y'' - 2xy - x^2 y'$$
$$= (x+1)y'' + (1-x^2)y' - 2xy$$
$$y'''(0) = (0+1)\underbrace{y''(0)}_{1} + (1-0^2)\underbrace{y'(0)}_{1} - 2(0)\underbrace{y(0)}_{1}$$
$$= 2$$

$\boxed{y'''(0) = 2}$

Differentiate the y''' formula above to get:

$$y^{(4)} = y'' + (x+1)y''' + (-2x)y' + (1-x^2)y''$$
$$-2y - 2xy'$$
$$y^{(4)}(0) = \underbrace{y''(0)}_{1} + (0+1)\underbrace{y'''(0)}_{2} - 2(0)\underbrace{y'(0)}_{1} + (1-0^2)\underbrace{y''(0)}_{1}$$
$$-2\underbrace{y(0)}_{1} - 2(0)\underbrace{y'(0)}_{1}$$
$$= 1 + 2 - 0 + 1 - 2 - 0$$
$$= 2$$

$\boxed{y^{(4)}(0) = 2}$

Thus,

$$y(x) = 1 + x + \frac{1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{2}{4!}x^4 + \cdots$$
$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x + \frac{1}{12}x^4 + \cdots$$

with radius of convergence $r = \infty$, so it converges for $-\infty < x < \infty$.

$\boxed{②}$ We want a power series solution for

$$y'' + \frac{x}{1-x^2}y' - \frac{1}{1-x^2}y = 0$$

$$y'(0) = 1, \quad y(0) = 1$$

Here $x_0 = 0$

We have that

$$a_1(x) = \frac{x}{1-x^2} = x + x^3 + x^5 + \dots$$

$$a_0(x) = \frac{-1}{1-x^2} = -1 - x^2 - x^4 - \dots$$

from previous HW we also saw in that HW that $r=1$ for both of these

$$b(x) = 0$$

$r = \infty$

The minimum of the above r is $r=1$.

Thus the initial-value problem will have a unique solution

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \dots$$

with radius of convergence $r=1$.

We have

$$\boxed{\begin{aligned} & y'' + \frac{x}{1-x^2} y' - \frac{1}{1-x^2} y = 0 \\ & y'(0) = 1 \\ & y(0) = 1 \end{aligned}}$$

$$\boxed{\begin{aligned} & y(0) = 1 \\ & y'(0) = 1 \end{aligned}}$$

So,

$$\boxed{\begin{aligned} & y''(0) + \frac{0}{1-0^2} \underbrace{y'(0)}_{1} - \frac{1}{1-0^2} \underbrace{y(0)}_{1} = 0 \\ & y''(0) - 1 = 0 \\ & y''(0) = 1 \end{aligned}}$$

$$\boxed{y''(0) = 1}$$

You could try differentiating $y'' = \frac{-x}{1-x^2} y' + \frac{1}{1-x^2} y$ to get y''', $y^{(4)}$, etc, but it get's complicated quickly. Instead multiply by $(1-x^2)$ and use this equation, so there are no fractions:

$$(1-x^2) y'' + x y' - y = 0$$

Now just differentiate the above:

$$(-2x)y'' + (1-x^2)y''' + y' + xy'' - y' = 0$$

$$(1-x^2)y''' - xy'' = 0$$

$$(1-0^2)y'''(0) - (0)\underbrace{y''(0)}_{1} = 0$$

$$y'''(0) = 0$$

$y'''(0)=0$

Now use $(1-x^2)y''' - xy'' = 0$ from above to get $y^{(4)}$. We have

$$(-2x)y''' + (1-x^2)y^{(4)} - y'' - xy''' = 0$$

$$(1-x^2)y^{(4)} - 3xy''' - y'' = 0$$

$$(1-0^2)y^{(4)}(0) - 3(0)\underbrace{y'''(0)}_{1} - \underbrace{y''(0)}_{1} = 0$$

$$y^{(4)}(0) = 1$$

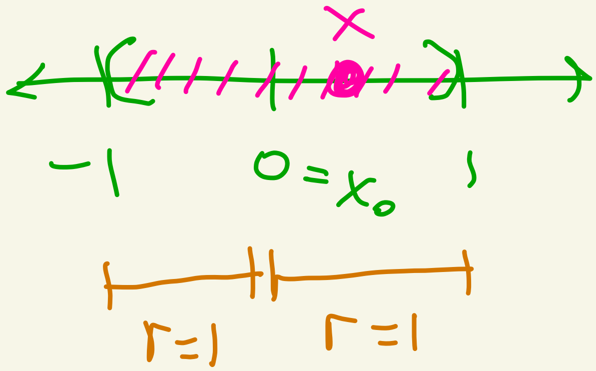
$y^{(4)}(0)=1$

So,

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \cdots$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots$$

with radius of convergence at least $r=1$ around $x_0 = 0$. So, it converges for at least $-1 < x < 1$.



③ We want a power series solution to the initial-value problem

$$\left.\begin{aligned} &xy'' + x^2 y' - 2y = 0 \\ &y'(1) = 1,\ y(1) = 1 \end{aligned}\right\}$$

Here $x_0 = 1$

Divide by x to get

$$y'' + xy' - \frac{2}{x} y = 0$$

Note that

$$a_1(x) = x = 1 + (x-1) \quad \leftarrow \quad r = \infty$$

$$a_2(x) = -\frac{2}{x} = -2\sum_{n=1}^{\infty} (-1)^{n+1}(x-1)^{n-1} \quad \leftarrow \quad r = 1 \text{ from class and HW}$$

$$b(x) = 0 \quad \leftarrow \quad r = \infty$$

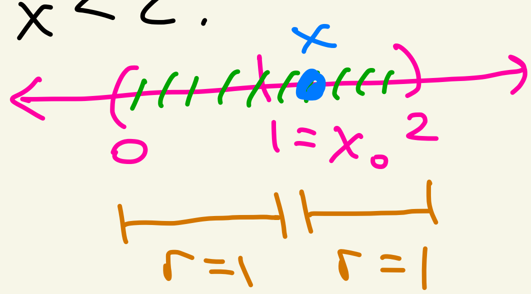
The minimum for the above r is $r = 1$.

Thus, the initial-value problem has a power series

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(1)}{n!}(x-1)^n$$

with at least radius of convergence $r = 1$.

So, it will converge for $0 < x < 2$.



Let's find $y(x)$.

We have:

$$y'' + x y' - 2x^{-1} y = 0$$

$$y'(1) = 1, \ y(1) = 1$$

$$y''(1) - 1 \cdot \underbrace{y'(1)}_{1} - 2(1)^{-1} \underbrace{y(1)}_{1} = 0$$

$$y''(1) - 3 = 0$$

$$y''(1) = 3$$

$y(1) = 1$
$y'(1) = 1$
$y''(1) = 3$

Differentiate $y'' + xy' - 2x^{-1}y$ to get the next step.

$$y''' + y' + xy'' + 2x^{-2}y - 2x^{-1}y' = 0$$

$$y''' + xy'' + (1 - 2x^{-1})y' + 2x^{-2}y = 0$$

$$y'''(1) + 1 \cdot \underbrace{y''(1)}_{3} + (1 - 2(1)^{-1}) \underbrace{y'(1)}_{1} + 2(1)^{-2} \underbrace{y(1)}_{1} = 0$$

$$y'''(1) + 3 - 1 + 2 = 0 \longrightarrow y'''(1) = -4$$

$y'''(1) = -4$

Differentiate the y''' formula above to find a formula for $y^{(4)}$.

We get

$$y^{(4)} + y'' + xy''' + (2x^{-2})y' + (1-2x^{-1})y'' - 4x^{-3}y + 2x^{-2}y' = 0$$

$$y^{(4)} + xy''' + (2-2x^{-1})y'' + 4x^{-2}y' - 4x^{-3}y = 0$$

$$y^{(4)}(1) + (1)\underbrace{y'''(1)}_{-4} + (2-2(1)^{-1})\underbrace{y''(1)}_{3} + 4(1)^{-2}\underbrace{y'(1)}_{1} - 4(1)^{-3}\underbrace{y(1)}_{1} = 0$$

$$y^{(4)}(1) - 4 + 0 + 4 - 4 = 0$$

$$y^{(4)}(1) = 4$$

$$\boxed{y^{(4)}(1) = 4}$$

Thus, for $-1 < x < 1$ we have

$$y(x) = y(1) + y'(1)(x-1) + \frac{y''(1)}{2!}(x-1)^2 + \frac{y'''(1)}{3!}(x-1)^3 + \frac{y^{(4)}(1)}{4!}(x-1)^4 + \cdots$$

$$= 1 + (x-1) + \frac{3}{2!}(x-1)^2 - \frac{4}{3!}(x-1)^3 + \frac{4}{4!}(x-1)^4 + \dots$$

$$= 1 + (x-1) + \frac{3}{2}(x-1)^2 + \frac{2}{3}(x-1)^3 + \frac{1}{6}(x-1)^4 + \dots$$

$3! = 3 \cdot 2 \cdot 1 = 6$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

(4) We want a power series solution to the initial value problem

$$y'' + \sin(x) y' + e^x y = 0$$
$$y'(0) = 1, \quad y(0) = 1$$

Here $x_0 = 0$

We have

$$a_1(x) = \sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots$$
$$a_0(x) = e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$$
$$b(x) = 0$$

these all have $r = \infty$

Thus, the initial-value problem has a solution

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n$$
$$= y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \dots$$

that converges for $-\infty < x < \infty$.

Let's find $y(x)$.

We have:

$$\boxed{\begin{aligned} &y'' + \sin(x)y' + e^x y = 0 \\ &y'(0)=1,\ y(0)=1 \\ &y''(0) + \underbrace{\sin(0)}_{0}\underbrace{y'(0)}_{1} + \underbrace{e^0}_{1}\underbrace{y(0)}_{1} = 0 \\ &y''(0) = -1 \end{aligned}}$$

$\boxed{y''(0) = -1}$

Differentiate $y''+\sin(x)y'+e^x y = 0$ to find y'''. We get

$$\boxed{\begin{aligned} &y''' + \cos(x)y' + \sin(x)y'' + e^x y + e^x y' = 0 \\ &y''' + \sin(x)y'' + (\cos(x)+e^x)y' + e^x y = 0 \\ &y'''(0) + \sin(0)y''(0) + (\cos(0)+e^0)y'(0) + e^0 y(0) = 0 \\ &y'''(0) + 0\cdot(-1) + (1+1)(1) + (1)(1) = 0 \\ &y'''(0) = -3 \end{aligned}}$$

$\boxed{y'''(0) = -3}$

So,

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \dots$$

$$= 1 + x + \frac{-1}{2!}x^2 - \frac{3}{3!}x^3 + \dots$$

$$= 1 + x - \frac{1}{2}x^2 - \frac{1}{2}x^3 + \dots$$

for $-\infty < x < \infty$.